# A Primer on Bond Math, Part I

## Finite payment structure can make bonds challenging to analyze.

QUANT U

Paul D. Kaplan

The two staples of any diversified portfolio are stocks and bonds. What makes these two asset classes differ is not only in the types of claims they represent (residual ownership versus debt obligation), but also in the structure of their payments to investors. Whereas stocks are perpetual and can make dividend payments that are unknown ahead of time, bonds exist for finite periods of time, during which they make payments that are specified in advance. It is this finite payment structure that makes analyzing bonds somewhat complex. In this issue of Quant U, I explain the foundations of bond analysis.

**Zero-Coupon Bonds**

The most elementary type of bond is a zero-coupon bond that is free of default risk. This type of bond makes a single payment on its maturity date in the future. It is default-free if it is issued by a sovereign government that will never default on its debt, such as the federal governments of the United States, Germany, and Canada. Throughout this issue of Quant U, the bonds I discuss are default-free.

In *principle*, the prices of bonds are determined in the bond market. Hence, we can take the prices of zero-coupon bonds as given. However, because the longer the time to maturity, the lower the price, it is hard to make meaningful price comparisons of market prices. To take into account the time-to-maturity of a zero-coupon bond, we calculate the yield-to-maturity or interest rate of each bond. These can be meaningfully compared across bonds of different maturities.

There are two main ways to express the yield on a bond; in continuous time or in discrete time.

In continuous time, we have:

$$Z(T) = e^{-s(T)T}$$

Where:

$Z(T)$ = the price of a coupon bond that pays one unit of money $T$ years in the future

$s(T)$ = the continuous-time spot rate for zero-coupon bonds that mature in $T$ years

Therefore, to derive the continuous-time spot rate for a zero-coupon bond that matures in $T$ years from its price, use the following formula:

$$s(T) = -\frac{\ln[Z(T)]}{T}$$

The price of the zero-coupon bond can also be expressed in terms of its discrete-time spot yield as follow:

$$Z(T) = \left(1 + \frac{y_s(T)}{q}\right)^{-qT}$$

Where:

$y_s(T)$ = the discrete-time spot yield for a zero-coupon bond that matures in $T$ years

$q$ = the number of coupon payments per year of coupon bonds

The reason for adjusting for the number of coupon payments per year of coupon bonds, even though by definition a zero-coupon bond makes no coupon payments, is that we want to compare the yields on zero-coupon bonds with yields on coupon bonds.

To derive the continuous spot rate for a zero-coupon bond that matures in $T$ years from its price, use this formula:

$$y_s(T) = q\left(Z(T)^{-\frac{1}{qT}} - 1\right)$$

The discrete-time spot yield can be derived from the continuous-time spot rate as follows:

$$y_s(T) = q\left(e^{\frac{s(T)}{q}} - 1\right)$$

**Spot Yield Curves**

Earlier, I said that in *principle*, the prices of zero-coupon bonds are determined in the bond market. In *reality*, there are a limited number of zero-coupon and coupon-paying bonds that trade each time. Fortunately, there is a way around this limitation: by fitting a smooth curve to the yields that are observed in the market. Such a curve, called a yield curve or the term structure of interest rates, provides an estimate of the yield at every possible maturity.

There are several yield curve models that are used throughout the world, the two common ones being the Nelson-Siegel (1987) and Svensson (1994) models. Each of these models can capture the various possible shapes of the yield curve: rising, falling, humped, and S-shaped. The Nelson-Siegel can capture these shapes using the four parameters $\beta_0$, $\beta_1$, $\beta_2$, and $\tau$ as follows:

$$s(T) = \beta_0 + \beta_1 \frac{1-e^{-T/\tau}}{T/\tau} + \beta_2 \left(\frac{1-e^{-T/\tau}}{T/\tau} - e^{-T/\tau}\right)$$

The Svensson model extends the Nelson-Siegel model by adding two parameters that allow for a second hump in the yield curve. The model is:

$$s(T) = \beta_0 + \beta_1 \frac{1-e^{-T/\tau_1}}{T/\tau_1} + \beta_2 \left( \frac{1-e^{-T/\tau_1}}{T/\tau_1} - e^{-T/\tau_1} \right) + \beta_3 \left( \frac{1-e^{-T/\tau_2}}{T/\tau_2} - e^{-T/\tau_2} \right)$$

The European Central Bank, or ECB, is one of the central banks that use the Svensson model. Each trading day, the ECB estimates the six parameters in the above equation for AAA-rated bonds in the eurozone and maintain a historical record of the results. Using the parameter values for three particular days, I created a yield curve of continuous spot rates for each of these days using the above equation. I converted to discrete-time spot yields with $q=1$ and plotted the resulting curves. **EXHIBIT 1** shows the results for three dates.

**Forward Rates**

The spot yield curve is often decomposed into a forward rate curve. A forward rate is an interest rate that applies to a short period in the future. If we look at the spot yield curve at monthly intervals, we can derive a forward rate curve at monthly intervals in which each forward rate is for one month in the future. Once this done, we can decompose each spot yield into a series of forward rates as follows:

$$y_s(T) = q \left\{ \left[ \left(1+\frac{F(\Delta t)}{q}\right) \left(1+\frac{F(2\Delta t)}{q}\right) \cdots \left(1+\frac{F(T)}{q}\right) \right]^{\frac{\Delta t}{T}} - 1 \right\}$$

Where:

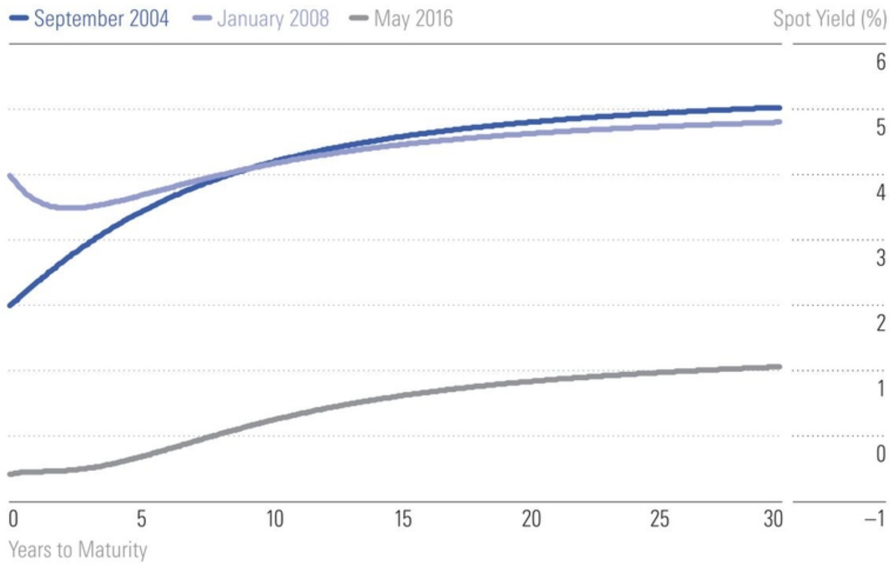
$\Delta t$ = the length of time of period of the forward rates in years, such as 1⁄12 being one month

$F(t)$ = the forward rate for the period $t - \Delta t$ through $t$

This decomposition of each spot yield into a series of forward rates gives rise to the *expectations hypothesis* of the yield curve. According to this theory, each spot yield is the expected return from buying a short-term bill, and rolling it over into another short-term bill, repeating this process until the zero-coupon bond matures. This leads

EXHIBIT 1

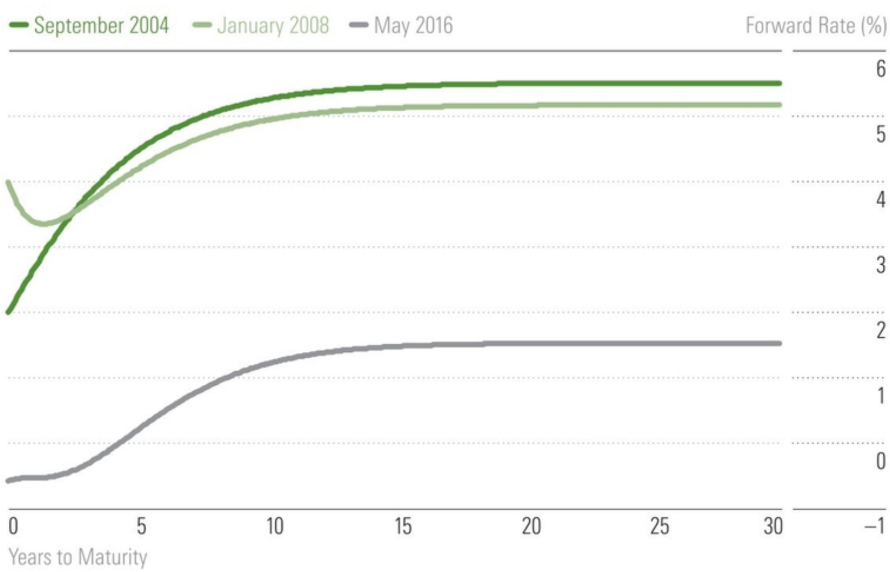
## Spot Yield Curves for AAA Eurozone Bonds




Source: European Central Bank, author's calculations

EXHIBIT 2

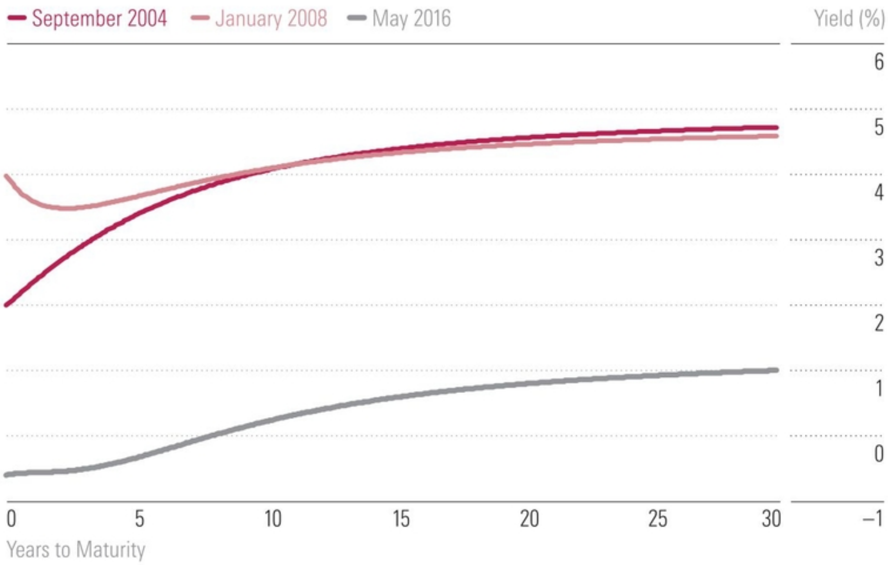
## Forward Rate Curves for AAA Eurozone Bonds




Source: European Central Bank, author's calculations

EXHIBIT 3

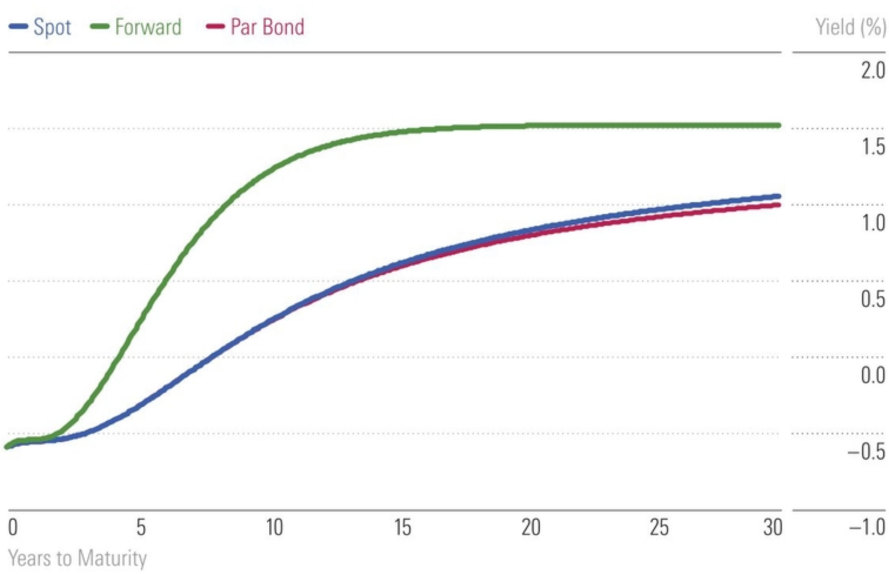
## Par Coupon Bond Yield Curves for AAA Eurozone Bonds




Source: European Central Bank, author's calculations

EXHIBIT 4

## Spot, Forward, and Par Coupon Bond Yield Curves for AAA Eurozone Bonds: May 2016




Source: European Central Bank, author's calculations

to the conclusion that each forward rate is the expected value of the interest rate on the bill that corresponds to its time period.

A forward rate curve can be derived directly from a spot yield curve by setting the first forward rate to the first spot yield, and then applying the following formula to obtain the rest of them:

$$F(T) = q\left\{\left[\frac{\left(1+\frac{y_s(T)}{q}\right)^{T}}{\left(1+\frac{y_s(T-\Delta t)}{q}\right)^{T-\Delta t}}\right]^{\frac{1}{\Delta t}} - 1\right\}$$

I used the formula to derive the forward rate curve from each of the spot rate curves in **EXHIBIT 1**. **EXHIBIT 2** shows the results for the same three dates as shown in **EXHIBIT 1**.

**Coupon Bonds**

Coupon bonds are the most common type of bond. A coupon bond makes regular payments, *coupons*, until the bond matures at which time it pays its *face value*.

*Pricing*

To price a coupon bond, we first need to find the dates of the coupon payments, starting with the first coupon. On the day that a bond is issued, the date of the first coupon payment is typically one coupon period in the future. This is one year if coupons are paid annually and six months if they are paid semiannually. But as bonds are traded on secondary markets, the date of sale can occur anytime during a coupon period.

Strictly speaking, we should find coupon payment dates precisely using a calendar. But for my purposes here, I present a reasonable approximation. First, we need to calculate the number of coupon payments. Letting $T$ denote the years to maturity of the bond, the number of payments is:

$$n = \lceil qT \rceil$$

where the brackets ⌈ ⌉ means to take the "ceiling" of the number. This means that if $x$ is an integer, $\lceil x \rceil$ is $x$, but if $x$ is not an integer, $\lceil x \rceil$ is the integer just higher than $x$. For example, $\lceil 3 \rceil = 3$, but $\lceil 3.1 \rceil = 4$. To see how the above formula works, suppose that we are looking at a 10-year bond with seminal annual coupons on the date of issue. Then $T$=10 and $q$=2 so that $n$=20. But if we are looking at a bond trading on a secondary market with 10 years and two months to maturity, so that $T$=10⅙ , $n$=21 because although the coupons are six months apart, the first one will be only two months from now.

When pricing a bond trading on a secondary market, we need to know what fraction of a coupon period (1/$q$) will pass until the first coupon is paid. We estimate this as follows:

$$f = n - qT$$

Note that the date of issue and on each coupon payment date, $f$=0.

Continuing our example, $f$=21–(2·10⅙)=⅔. This means that we will be two-thirds of a six-month period (four months) will have passed when the first coupon is paid.

We can separate the bond into two parts: the coupon payments and the face value of the bond. To do this, we define an annuity bond as one that pays one unit of money per year. This is a portfolio of zero-coupon bonds and is priced as follows:

$$A(T) = \frac{1}{q}\sum_{j=1}^{n} Z\left(\frac{j-f}{q}\right)$$

Let $c$ denote the annual coupon rate; i.e., the amount paid each year in coupons as a fraction of the bond's face value. So, if a bond has a face value of 100 and pays a semiannual coupon at a rate of 4% per year, each semiannual coupon is 2. For our purposes here, I present the formula for the price of a coupon bond per unit of face value. This is:

$$P(c,T) = c \cdot A(T) + Z(T)$$

In the bond world, this price is known as the *dirty price* of the bond. It is what the buyer of a bond pays the seller. But this is usually decomposed into two components: *accrued interest* and the *clean price*. The idea of accrued interest is that when the bond is purchased during a coupon period, the coupon partly belongs to the seller. The fraction that belongs to the seller is the accrued interest which is given by:

$$AI(c) = f\,\frac{c}{q}$$

The clean price is just the dirty price less the accrued interest:

$$CP(c,T) = P(c,T) - AI(c)$$

The buyer pays the seller the clean price plus the accrued interest. In bond trading, the clean price is quoted and the accrued interest is calculated. Here, I have presented an approximation for the fraction that belong to the seller in the accrued interest. In actual bond trading, this is calculated using a calendar with in conjunction with the day count convention for the type of bond.

*Coupon Bond Yields*

The yield on a coupon bond is defined as the annualized internal rate of return of its payments given its price. In other words, it is the number $y$ that solves the following equation:

$$P(c,T) = \left(\frac{c}{q}\sum_{j=1}^{n}\left(1+\frac{y}{q}\right)^{-j} + \left(1+\frac{y}{q}\right)^{-n}\right)\left(1+\frac{y}{q}\right)^{f}$$

This equation can be written as:

$$P(c,T) = \left(\frac{c}{y}\left[1-\left(1+\frac{y}{q}\right)^{-n}\right] + \left(1+\frac{y}{q}\right)^{-n}\right)\left(1+\frac{y}{q}\right)^{f}$$

A coupon bond is priced at par if its yield is equal to it coupon. This makes its price:

$$P(y,T) = \left(1+\frac{y}{q}\right)^{f}$$

To create a yield curve for coupon bonds at par, we solve the above equation for $y$ at each maturity. I did this using the yield curve data from the ECB. **EXHIBIT 3** shows the results for the same three dates as shown in **EXHIBITS 1** and **2**. If the bond is issued at par, $f$=0 on the date of issue its priced at face value and the yield is:

$$y = \frac{1-Z(T)}{A(T)}$$

The spot, forward, and par bond yield curves shown in **EXHIBITS 1**, **2**, and **3**, respectively, appear to be similar. To highlight the differences, I plot the three kinds of yield curves for the final date shown in the previous exhibits together in **EXHIBIT 4**.

Now that we have a foundation for the math of bonds, we will expand our bond analysis in future issues of Quant U. 

Paul D. Kaplan, Ph.D., CFA, is director of research with Morningstar Canada. He is a member of the editorial board of Morningstar magazine. The author thanks Tom Teder of Morningstar Canada for his helpful edits and comments.

References

Nelson, Charles R. and Andrew F. Siegel. 1987. "Parsimonious Modeling of Yield Curves." *Journal of Business*, 60 (October), pp. 474–489.

Svensson, Lars E.O. 1994. "Estimating and Interpreting Forward Interest Rates: Sweden 1992–1994." Cambridge: National Bureau of Economic Research, Working paper 4871.