A Primer on Bond Math, Part III

How to consider bonds' unique features when calculating rates of return.

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This is the last of a three-part series on bond math. In the first part (August/September 2016), I discussed the how default-free bonds are priced from market-determined yield curves and how from those yield curves we can derive the prices and yields on both zero-coupon bonds and coupon bonds that are traded in secondary markets. In the second part (October/November 2016), I showed how to the measure the risk of a coupon bond. In this final installment, I discuss how to take into account the unique features of zero-coupon bonds and coupon bonds when calculating rates of return on portfolios that contain them.

The most fundamental difference between a stock and a bond is that while a stock can, in principle, be held indefinitely, bonds mature and must be replaced to maintain a given strategy such as constant maturity or constant duration. To illustrate the consequence of this, I ran two Monte Carlo simulations, one on the price of a stock over a 30-year period and the other on the price of a 30-year coupon bond that is at par when purchased. **EXHIBIT 1** shows percentiles of the possible evolution of the stock price, and **EXHIBIT 2** shows the same for the bond price. In **EXHIBIT 1**, we see that over time, the spread between the 95th and 5th percentiles of the stock price grows. In contrast, while the spread between the 95th and 5th percentiles of the bond price grows for about the first 10 years, it gradually shrinks toward 0 out at 30 years where the future price of the bond is known with certainty.

Calculating Returns

Stocks

In general, the total return on a security over a period of time is the sum of its income return and its price return. For a stock, its income return is any income received by the investor over the period, which typically is a dividend, divided by the price at the end of the last period. Its price return is the change in the price over the period divided by the price at the end of the period. Mathematically,

$$IR_{St} = \frac{I_{St}}{P_{St-\Delta t}}$$

$$PR_{St} = \frac{P_{St}}{P_{St-\Delta t}} - 1$$

$$TR_{St} = IR_{St} + PR_{St}$$

where:

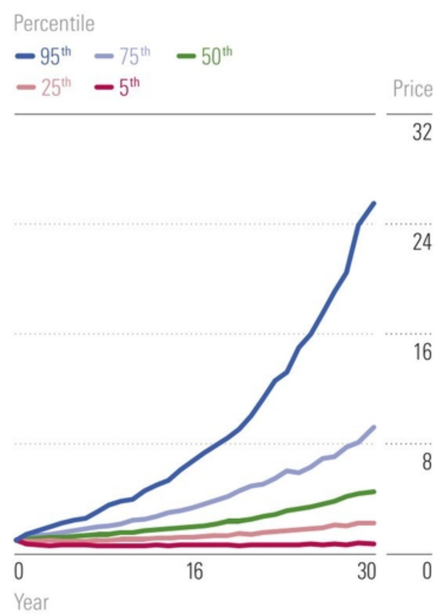
IR_{St} = income return of the stock for period t
PR_{St} = price return of the stock for period t
TR_{St} = total return of the stock for period t
I_{St} = income return of the stock for period t
P_{St} = price of the stock at the end of period t
Δt = the length of the period

Zero-Coupon Bonds

With bonds, it is a bit different. The income return on a bond is what the total return would be if the yield on the bond did not change from time $t-\Delta t$ to time t. The price return is simply the difference between the total return and the income return. To see how this works, consider the price of a zero-coupon bond with T years to maturity at time $t-\Delta t$:

EXHIBIT 1

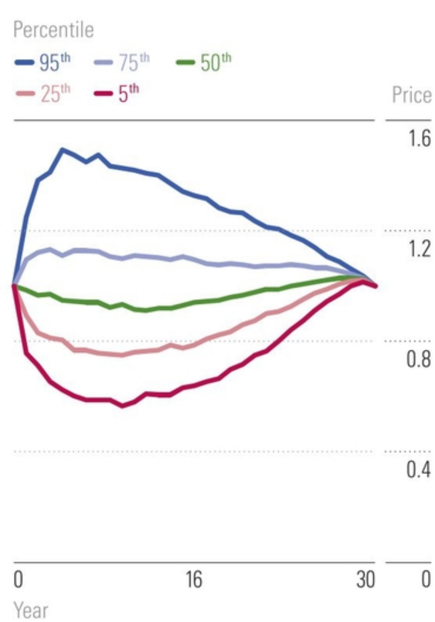
Simulated Stock Prices Over 30 Years



Source: Morningstar

EXHIBIT 2

Simulated Prices of a 30-Year Coupon Bond at Par Held to Maturity



Source: Morningstar

$$P_{Zt-\Delta t}\left(T; y_{st-\Delta t}(T)\right) = \left(1+\frac{y_{st-\Delta t}(T)}{q}\right)^{-qT}$$

where:

$P_{Zt-\Delta t}(T;.)$ = the price of a zero-coupon that matures in T years at the end of period $t-\Delta t$

$y_{st-\Delta t}(T)$ = the yield of a zero-coupon that matures in T years at the end of period $t-\Delta t$

q = the number of coupon payments that coupon bonds make per year (Used here so that yields on zero-coupon bonds are comparable with those of coupon bonds.)

The total return on the bond over the period $t-\Delta t$ to t is the percentage change in its price:

$$TR_{Zt}(T) = \frac{P_{Zt}\left(T-\Delta t; y_{st}(T-\Delta t)\right)}{P_{Zt-\Delta t}\left(T; y_{st-\Delta t}(T)\right)} - 1$$

The income return is what the total return would be if the yield did not change:

$$IR_{Zt}(T) = \frac{P_{Zt}\left(T-\Delta t; y_{st-\Delta t}(T)\right)}{P_{Zt-\Delta t}\left(T; y_{st-\Delta t}(T)\right)} - 1$$

With some algebra, we can show that this is:

$$IR_{Zt}(T) = \left(1+\frac{y_{st-\Delta t}(T)}{q}\right)^{q\Delta t} - 1$$

Hence, the income return is the yield at the end of the previous period, restated for the length of the investment period. It might seem odd for an instrument that makes no income payments to have an income return. However, in some jurisdictions (including the United States), holders of zero-coupon bonds must pay income tax on income return, even though the instrument itself does not generate the cash needed to pay the taxes.

The price return is the difference between the total return and the income return:

$$PR_{Zt}(T) = TR_{Zt}(T) - IR_{Zt}(T)$$

Coupon Bonds

Things get more complex with coupon bonds. Using the formula for the price of a coupon bond from Part 1 of this series, we can write the price of a coupon bond at time $t-\Delta t$ as follows:

$$P_{Bt-\Delta t}\left(c,T; y=y_{Bt-\Delta t}(c,T)\right) = \left(\frac{c}{y}\left[1-\left(1+\frac{y}{q}\right)^{-n(T)}\right] + \left(1+\frac{y}{q}\right)^{-n(T)}\right)\left(1+\frac{y}{q}\right)^{f(T)}$$

where:

$P_{Bt-\Delta t}(c,T;.)$ = the price of a coupon bond with annual coupon rate c that matures in T years at the end of period $t-\Delta t$

$y_{Bt-\Delta t}(c,T)$ = the yield on a coupon bond with annual coupon rate c that matures in T years at the end of period $t-\Delta t$

$n(T)$ = the number of coupon payments that the bond will make past the end of period $t-\Delta t$

$f(T)$ = the fraction of a coupon payment that will have passed when the upcoming coupon payment is made. $f(T)=0$ on the day of issue and on all coupon payment dates.

When dealing with coupon bonds, it is important to understand the relationship between T, $n(T)$, and $f(T)$. To illustrate, let us ignore that fact that different calendar months have different numbers of days and treat all months as being one twelfth of a year. So, if a bond has 10 years and two months to maturity and makes semiannual coupon payments ($q=2$), we have:

- $n(T)=20$ because there are 20 coupon payments
- $f(T)= 2/3$ because when the next coupon payment is made, it will be four months into a six-month payment period, which is two thirds of the period

There are two cases to consider when calculating total return and income return:

1 No coupon is paid during the investment period.
2 There is a coupon paid during the investment period.

To keep things simple, I assume that the length of the coupon period is multiple of the length of the investment period. For example, with semiannual coupons, the length of coupon period is six months. With an investment period of one month ($\Delta t=1/12$), there are six investment periods in each coupon period. Under this assumption, at the end of an investment period that is also the end of a coupon period, $f(T)=0$ and $n(T)=n(T-\Delta t)-1$.

If T is not a coupon payment date, the total return is:

$$TR_{Bt}(c,T) = \frac{P_{Bt}\left(c,T-\Delta t; y_{Bt}(c,T-\Delta t)\right)}{P_{Bt-\Delta t}\left(c,T; y_{Bt-\Delta t}(c,T)\right)} - 1$$

If T is a coupon payment date, the total return is:

$$TR_{Bt}(c,T) = \frac{\frac{c}{q} + P_{Bt}\left(c,T-\Delta t; y_{Bt}(c,T-\Delta t)\right)}{P_{Bt-\Delta t}\left(c,T; y_{Bt-\Delta t}(c,T)\right)} - 1$$

The income return when is no a coupon payment is:

$$IR_{Bt}(c,T) = \frac{P_{Bt}\left(c,T-\Delta t; y_{Bt}(c,T)\right)}{P_{Bt-\Delta t}\left(c,T; y_{Bt-\Delta t}(c,T)\right)} - 1$$

With some algebra, we can show that whether or there is a coupon payment, the income return is:

$$IR_{Bt}(c,T) = \left(1+\frac{y_{Bt-\Delta t}(c,T)}{q}\right)^{q\Delta t} - 1$$

Note that is the same formula for the income return on a zero-coupon bond. Hence, the income return is the yield at the end of the previous period, restated for the length of the investment period. The price return is the difference between the total return and the income return:

$$PR_{Bt}(c,T) = TR_{Bt}(c,T) - IR_{Bt}(c,T)$$

Historical Total Returns

Using the yield-curve data from the European Central Bank and the techniques that I described in Part I of this series, and the formulas I present here, I formed a history of monthly total returns on zero-coupon bond and par coupon bonds over the 12-year period October 2004 to September 2016. I use 60 maturities from half a year to 30 years. In each month, after each bond matures by one month, I replace it with a bond of the fixed maturity. In other words, I model constant maturity strategies. To see how historical risk and return vary by maturity, for each of the 60 fixed maturities, I calculate the compound annual return and annualized standard deviation. **EXHIBIT 3** presents the results.

From **EXHIBIT 3**, we see that risk increases with maturity. This is as should be expected since as I discussed in Part II of this series, the sensitivity of a bond's price to changes in its yield, its duration, increases with maturity. Because zero-coupon bonds have greater duration than coupon bonds, the risk is greater. In **EXHIBIT 3**, this becomes apparent for maturities greater than 10 years.

EXHIBIT 3 also shows that over this period, investors who followed a fixed maturity strategy would have been rewarded with greater returns the greater the risk taken. Notice that not only are the returns greater the greater the maturity, but zero-coupon bonds, with their greater risk, delivered greater returns than their coupon bond counterparts. This is not to say that greater risk will always result in greater returns (especially in the current low interest rate environment), just that was the case historically.

Approximating the Price Return with Duration and Convexity

In Part II of this series, I introduced the concepts of duration and convexity and how they can be used to estimate the percentage change in price in response to change in yield. The approximation formula is:

$$\frac{\Delta P}{P} \approx -D_m \Delta y + \frac{1}{2} C (\Delta y)^2$$

where:

D_m = the modified of the bond

C = the convexity of the bond

To see how well this formula works, I use it to approximate the price returns for the fixed maturity strategies summarized in **EXHIBIT 3**. In general, the greater the maturity, the greater the error. But even at the highest maturity which I consider, 30 years, most of the errors are small. For 30-year par coupon bonds, in 41 of the 144 months considered, the error rounds to 0.00%. For 30-year zero coupon bonds, in 43 of the 144 months, the error rounds to 0.00% For the par coupon bonds, in 127 months, the errors are no more than 0.1% and for the zero-coupon bonds, this is the case in all but five months. The largest errors occur in August 2010 when the yield on par coupon bonds fell by 0.72% and the yield on zero-coupon bonds by 0.8%. In that month, the errors were 0.57% and 0.21% on par coupon and zero-coupon bonds, respectively.

Tip of the Iceberg

In this three-part series, I have presented the basic analytics on "plain vanilla" high-grade bonds, including yield curves, risk measures, and performance measures. While these analytics may seem complicated, they're only the tip of the iceberg of fixed-income analytics. Bonds can take many shapes and forms by (1) being of lower credit quality and thus requiring credit analysis, and (2) by embodying various option features such as being callable. Fortunately, there are a number fine books on fixed-income analysis that investors can study to get into these more advanced topics. M

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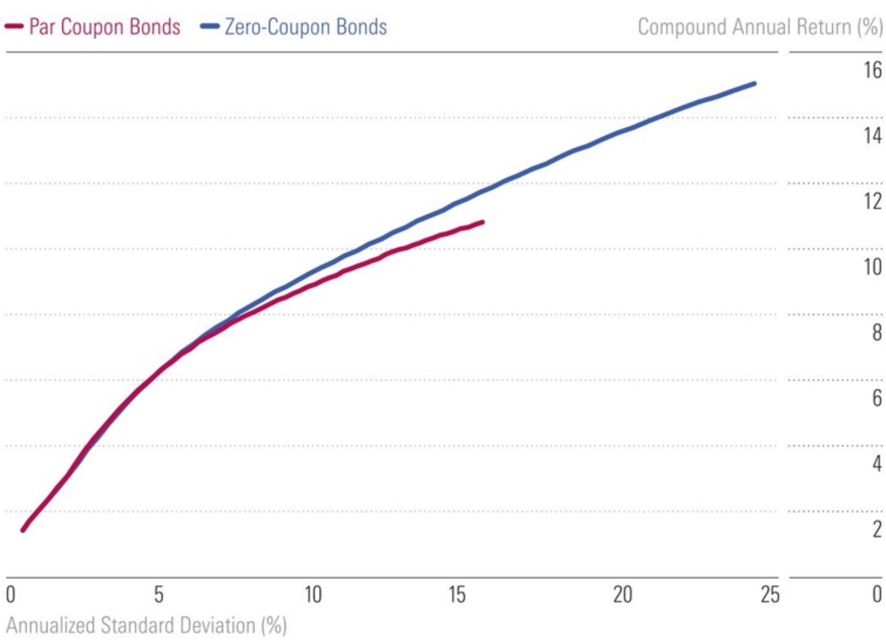
EXHIBIT 3

Historical Risk and Returns on Fixed Maturity Strategies: October 2004–September 2016



Source: European Central Bank, author's calculations